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A BASIS OF EQUIVALENCE CLASSES OF PATHS IN OPTIMIZATION PROBLEMS

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UDC 514+517

Development of control systems and optimum-seeking procedures often leads to problems that involve finding extrema of functions on paths in regions that are not simply connected. The construction of a complete and sufficiently simple mathematical description of the space of paths that will allow using more efficiency optimization methods than those currently available remains, however, an open question.

The existing methods for the solution of such problems (e.g., tracing problems [1]) involve enumeration and optimization of paths from a certain set defined as a set of routes on a graph [1, 2] or as a set of equivalence classes of paths [3]. In the first group of methods, optimization of a given path on a graph amounts to evaluating the objective function, while in the second group we seek an optimal path on a given equivalence class. The difference between the two methods is best illustrated by the following example. A crossing path on a graph (ABCDECFG, Fig. 1) may define an equivalence class which includes a minimum-length path without a crossing (AG in Fig. 1), and conversely a noncrossing path on a graph (ABCDE, Fig. 2) may define an equivalence class in which the minimal-length path ($AB_1C_1D_1E$ in Fig. 2) has a crossing.

Thus, before we can approach the solution of the corresponding optimization problem on a path space, we have to construct an exact mathematical model of the space. To this end, we introduce the notation of a basis of equivalence classes of paths in the set F homeomorphic to a disk with $n > 0$ holes [4]. We prove the existence of such a basis and derive its properties which permit:

– deciding questions of completeness of various path spaces;

– constructing a sufficiently useful and simple mathematical description of the set F;

– solving optimization problems by constructing appropriate deformations in F (see, e.g., [5]).

The set F admits the following representation:

$$F=[D_0]\setminus\left(\bigcup_{i=1}^{n} D_i\right);$$

$$[D_i]\cap[D_j]=\varnothing\ (i\neq j;\ i, j=1, 2, \dots, n),$$

$$[D_i]\subset D_0\ (i=1, 2, \dots, n),$$

where D_i ($i = 0, 1, \dots, n$) are compact sets in R_2 homeomorphic to the open circle, and $[D_i]$ are homeomorphic to the closed circle.

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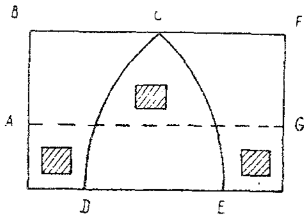

Fig. 1

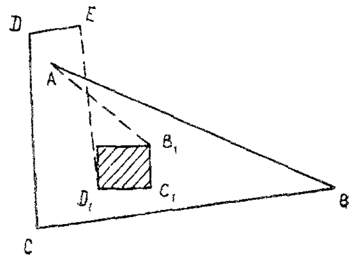

Fig. 2

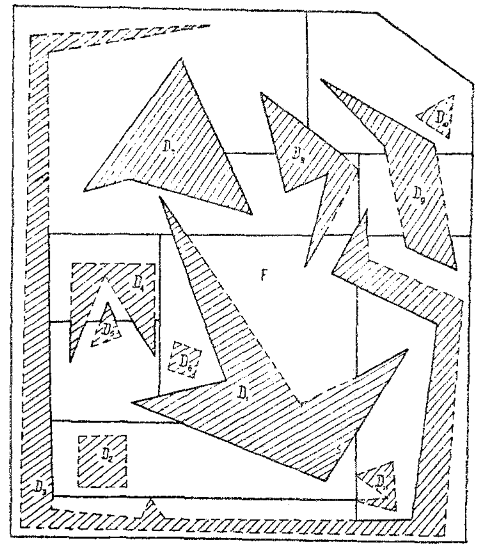

Fig. 3

Following the terminology of [6], a net $\Gamma = \{\gamma_i\}_{i=1}^m$ in F is any finite collection of noncrossing continuous curves of bounded length which lie in F and have no common points, with the possible exception of the endpoints. The curves γ_i ($i = 1, 2, \ldots, m$) are called edges, and their ends are called vertices.

In what follows, we only deal with connected nets.

The curves γ_i ($i = 1, 2, \ldots, m$) may be considered as continuous mappings of closed intervals into F. In this sense we say that the net Γ generates a certain set of paths $G = \{g_i\}_{i=1}^m$. Following [4], we denote by pq the product of the paths p and q; p^{-1} is the inverse of the path p; $p \sim q$ denotes equivalence of the paths p and q in F. Furthermore, let T_1 and T_2 be the initial and terminal points of some path p, then we say that p joins T_1 and T_2. If $T_1 = T_2 = T$, the path p is called a loop with base point T (or T-based loop) [4].

Let g (indexed or not) denote a path in G or the inverse of some path in G. Thus, if the path p is the product $g_1 g_2 \ldots g_k$, the sequence $(g_1, g_2, \ldots, g_k)$ is called a decomposition of the path p in G and we say that the path p is generated by the set G. Note that the decomposition of the path p may include some repeating paths from G.

Consider a net Γ in F which generates the set G. For any point $T \subset F$ we denote by $p_T \subset F$ an arbitrary fixed path joining the point T with an arbitrary vertex of the net Γ.

Definition. The set G forms a basis of equivalence classes of paths in F (more briefly, a basis) if for every two points ($A, B \in F$) and the corresponding paths (p_A, p_B) the following properties hold:

a) completeness – for every path $\tau \subset F$ joining A and B, there is an equivalent path $p_A p_* p_B^{-1}$, where p_* is a path generated by the set G;

b) minimality – no proper subset of edges of the net Γ generates a set of paths satisfying the completeness property.

To establish correctness of this definition, let us construct one of the possible bases. A deformation of a disk with $n > 0$ holes, in particular, is a n-leaf rose, i.e., a union of n topological circles touching only at some point Z, called the base point. Each of these circles partitions R_2 into a simply connected region containing one and only one of the regions D_i ($i = 1, 2, \ldots, n$) and an unbounded region. Under appropriate parametrization these circles define certain Z-based loops $\{h_i\}_{i=1}^n = H$ such that any loop $p \subset F$ with base point Z is equivalent to some product of loops from H. Moreover, for any proper subset H' of loops from H there is a Z-based loop which is not equivalent to any product of loops from H'.

Since H contains a single vertex, the corresponding paths p_A, p_B connect the points A and B with Z. Consider an arbitrary path $\tau \subset F$ joining A and B, then the path $p = p_B^{-1}\tau^{-1}p_A$ is a Z-based loop and there is a product p_* of loops from H such that $p \sim p_*$, i.e., the path $pp_*^{-1} = (p_B^{-1}\tau_-^{-1}p_A)p_*^{-1}$ is a unit loop in F, and so the paths $p_A p_*^{-1} p_B^{-1}$ and τ are equivalent. Therefore, the set H has the completeness property of a basis. Setting $A = B = Z$, we see that H also has the minimality property of a basis, since constant paths may be taken as p_A, p_B. Thus, H is a basis.

Clearly, a n-leaf rose is not the only topological type of a net generating a basis of equivalence classes of paths in F. It is thus interesting to derive conditions under which the net Γ defines a basis in F. The answer to this question is provided by the following theorem.

THEOREM 1. A connected net $\Gamma \subset F$ generates a basis of the equivalence classes of the paths in F if and only if it has no edges with a free endpoint and partitions R_2 into n + 1 regions $\{Q_i\}_{i=0}^{n}$, where Q_0 is unbounded, and $Q_1, Q_2, \ldots, Q_n$ are bounded regions such that

$$D_i \subset Q_i \ (i = 1, 2, \ldots, n). \tag{1}$$

Proof. Sufficiency. Denote by G the set of paths generated by the net Γ. We will prove that this is a basis.

Since Γ is connected, the regions Q_i (i = 1, 2, . . . ,n) are simply connected. Therefore on the boundary of the region Q_i there is a loop a_i generated by the set of paths G whose order is zero with respect to points from $R_2 \setminus [Q_i]$ and nonzero for points from Q_i (and therefore also from D_i). Each loop a_i (i = 1, 2, . . . , n) includes some vertex $Z_i \in \Gamma$. Take an arbitrary vertex Z of the net Γ and some paths p_i generated by the set G which join the points Z and Z_i. Then the loops a_i and $h_i = p_i a_i p_i^{-1}$ are homotopic in F and have the same order with respect to points from Q_j and $R_2 \setminus \Gamma$, respectively. This means that any product of Z-based loops from $H = \{h_i\}_{i=1}^{n}$ which does not contain the loop h_i (or h_j^{-1}) has zero order with respect to points from Q_j, and therefore cannot be equivalent to the loop h_j. On the other hand, from any collection containing n + 1 loops with base point Z, we can extract n loops whose various products generate all the equivalence classes of Z-based loops. Thus, the family H is homotopic in F to a n-leaf rose, and the set G has the completeness property of a basis.

Removing from the net Γ an edge γ will produce a disconnected net $\Gamma \setminus \gamma$, or will cause some two regions Q_i and Q_j ($i \neq j$) to merge into one [6]. The former is inadmissible. Suppose that the latter takes place. If one of these regions, say Q_i, is unbounded, the order of any loop on $\Gamma \setminus \gamma$ is zero with respect to points from D_j. If i, $j \neq 0$, then any loop on $\Gamma \setminus \gamma$ has the same order with respect to points from D_i and D_j. Assuming that the points A, B. and Z coincide and taking constant paths as p_A, p_B, we conclude that the completeness property is not satisfied by the set of paths G' generated by the net $\Gamma \setminus \gamma$. Therefore G also has the minimality property of a basis.

Necessity. Let the net Γ generate a basis G. Clearly, the net Γ contains no edges with a free endpoint (their deletion does not affect the completeness property for the set of paths generated by the remaining part of the net).

Being a connected net consisting of a finite number of finite-length edges, Γ partitions R_2 into one unbounded region Q_0 and a certain number m ($m \geq 0$) of bounded simply connected regions $Q_1, Q_2, \ldots, Q_m$. Repeating the same argument as in our proof of sufficiency, we find that $m \geq n$ and different regions D_i (i = 1, 2, . . . , n) are included in different regions Q_j (j = 1, 2, . . . , m). If m = n, the proof of necessity is complete.

Let m > n. Then the regions Q_j (j = 1, 2, . . . , m) may be partitioned into two subsets:

– the regions containing one of the D_i (i = 1, 2, . . . , n);

– the regions contained in F.

The first subset includes n regions, and the second m – n regions. Consider the following procedure deleting edges from a connected net, where Q' and Q'' ($Q' \cap Q'' = \emptyset$) are the two regions into which R_2 is partitioned by this net.

Delete the edge which is a common boundary (or part of the boundary) of two regions, Q' and Q'', one of which is contained in F (i.e., belongs to the second subset). Then delete all those edges which have acquired a free endpoint.

Applying this procedure to the net Γ, we obtain a net Γ_1, then apply the same procedure to the net Γ_1, and so on, until we obtain the net Γ_{m-n}. The application of this procedure to a connected net $\Gamma_i \subset F$ (i = 0, 1, . . . , m – n – 1; $\Gamma_0 = \Gamma$) clearly leaves the net Γ_{i+1} connected and does not break condition (1) for the regions into which Γ_{i+1} partitions R_2. Therefore the net Γ_{m-n} satisfies the condition of the theorem and thus generates a basis. Thus, for m > n, the net Γ does not satisfy the minimality condition of a basis.

We have thus proved necessity. QED.

From Theorem 1 we obtain the following properties of the net Γ and the generated basis G of equivalence classes of the paths in F.

Property 1. No path $g \in G$ (or g^{-1}) is equivalent in F to a product of paths from G which does not contain the path g (or g^{-1}).

Property 2. The net Γ is a deformation retract of the set F.

Property 3. Augmenting the basis G with any collection of paths does not break its completeness property.

Let E(A, B) be the set of equivalence classes of the paths in F that join the points A and B, and E(Z) the set of equivalence classes of Z-basis loops in F.

Property 4. For any two points A, B $\in$ F, the set E(A, B) is countable.

Proof. The set E(Z) is countable, independently of the choice of the basis point Z in F (up to homotopic equivalence) [4], and is generated by the set G. Then the paths p_A and p_B contracting the points A and B with some vertex Z $\in \Gamma$ define a one-to-one correspondence between the sets E(A, B) and E(Z). Therefore E(A,B) is also countable.

The path p generated by the basis G is called simple if its decomposition $(g_1, g_2, \ldots, g_m)$ contains no paths such that

$$g_i = g_{i+1}^{-1} \ (i = 1, 2, \ldots, m-1). \tag{2}$$

THEOREM 2. The simple paths generated by the basis G are equivalent in F if and only if their decompositions coincide.

Proof. Sufficiency is obvious. Let us prove necessity.

Let the basis G generate the simple paths p, q which are equivalent in F. Then their initial and terminal points coincide, defining a loop $e = pq^{-1}$, which is the unit loop in F. By Properties 1 and 2, the unit loop is contractable on Γ, and therefore the decomposition of the loop e contains at least one pair of paths from G for which (2) holds. This means that the decompositions of the paths p and q in the basis G coincide on the first or the last component paths from G. The remaining parts of the paths p and q are equivalent and for them (2) clearly holds. Thus, the decompositions of the paths p and q coincide.

Thus, for any pair of points A, B $\in$ F and arbitrary fixed paths P_A, $P_B \subset$ F joining these points with some vertices of the net Γ, the set of paths $\{G, p_A, p_B\}$ generates a countable set of all equivalence classes of the paths E(A, B) in F. To different simple paths on Γ correspond different equivalence classes.

The above results have led to a method for the solution of a number of shortest path problems for a given nonsimply connected polygonal region. These methods were realized in an application software package intended for the solution of tracing optimization problems [5]. Fairly high efficiency of the method has been ensured by applying an automatically constructed topological model of the path space (i.e., of the basis of equivalence classes of the paths in F) and also by seeking the solution directly in a linearly connected set F. The package uses the following algorithm (one of many possible algorithms) to construct the net Γ generating the basis G in F.

Algorithm

1. In each region D_i (i = 1, 2, . . . , n) fix an arbitrary point T_i.

2. Partition R_2 by the connected net $\Gamma \subset D_0$ into n + 1 regions, so that each of the bounded regions contains one and only one point T_i (i = 1, 2, . . . , n).

3. Set $\Gamma = \Gamma \bigcup_{i=1}^{n} L_i$, where L_i are the boundaries of the regions D_i.

4. Delete from Γ all the edges contained in D_i (i = 1, 2, . . . , n).

5. Apply to the net Γ the edge deletion procedure described in our proof of necessity of Theorem 1, until the number of bounded regions into which R_2 is partitioned by the net becomes equal to n.

Figure 3 shows a net Γ generated by a computer with the aid of this algorithm. The edges of the net Γ are traced by thick lines, and the regions D_i (i = 1, 2, . . . , n) are shaded.

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MEAN-OPTIMALITY PRINCIPLE FOR SYSTEMS WITH RANDOM JUMP STRUCTURE

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Systems with random jump structures were considered in [1] and elsewhere. In this article, we consider optimization of systems with a Poisson stream of smooth discontinuities. The principle of mean optimality is established for such systems as a natural extension of the principle of maximum.

The mathematical models of this class may be used to represent, for example, programmed control systems functioning in little known environments subject to random impulse forces, disturbance streams, sudden failures of some of the system elements leading to random jumpwise changes in system structure and parameters, optimization of pursuit and capture systems, optimization of aircraft impulse control systems with failing elements and environmental random impulse forces, etc. Here, as a rule, the number of the arrival time and the impulses are random variables, assumed to have a Poisson distribution.

1. STATEMENT OF THE PROBLEM

Consider a controllable [2] system

$$\begin{gathered} dX_i^k = \varphi_i^j(t, X^k, u, a)\,dt + \sum_{\nu=1}^{n} \sigma_{i\nu}^j(t, X^k)\,d\eta_\nu(t), \\ X_i^k(t_0) = X_{i0},\ t \in [T_j, T_{j+1}],\ T_0 = t_0,\ T_{k+1} = t_f \\ (i = 1, \ldots, n,\ j = 0, \ldots, k,\ k = 0, \ldots, k_0), \end{gathered} \tag{1.1}$$

in which the sequence of discontinuities on $[t_0, t_f]$ forms a stationary or a nonstationary Poisson stream of events [3]:

$$\begin{gathered} p_k = \frac{\lambda^k (t_f - t_0)}{k!} \exp(-\lambda(t_f - t_0)), \\ p_k = \frac{1}{k!}\left(\int_{t_0}^{t_f} \lambda\,dt\right)^k \exp\left(-\int_{t_0}^{t_f} \lambda\,dt\right) \quad (k = 0, \ldots, k_0), \end{gathered} \tag{1.2}$$

where p_k is the probability of k discontinuities on $[t_0, t_f]$, λ is the stream density (a known function of time).

The effectiveness of the control $v = (u, a)$ of system (1.1) is measured by the minimum of the functional

$$I_0(v) = M[\Phi_0(X_f, a) \mid k \leqslant k_0], \tag{1.3}$$

and the control objectives are defined by the relationships

$$\begin{gathered} I_s(v) = M[\Phi_s(X_f, a) \mid k \leqslant k_0] = 0 \quad (s = 1, \ldots, q), \\ I_s(u) = M[f_s(u) \mid k \leqslant k_0] = 0 \quad (s = 1, \ldots, q_0), \\ g_s(a) = 0 \quad (s = 1, \ldots, r_0). \end{gathered} \tag{1.4}$$

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